14 Power series

From now on I will consider not the general functional series, but a special, albeit highly important, class of them: power series.

14.1 Theory

Definition 14.1. *A power series with the center at $z = z_0$ is the expression of the form*

$$c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots + c_n(z - z_0)^n + \ldots = \sum_{n=0}^{\infty} c_n(z - z_0)^n, \tag{14.1}$$

where c_n are given complex numbers and the convention $0^0 = 1$ is used.

Example 14.2. Due to its importance, I recall our main example — geometric series, which is clearly a power series with the center at $z = 0$:

$$1 + z + z^2 + z^3 + \ldots + z^n + \ldots = \frac{1}{1 - z},$$

where the series converges absolutely, and hence pointwise, for all $|z| < 1$, and diverge for $|z| \geq 1$. Moreover, this series converges uniformly in each compact disk $\overline{B}(0, r) = \{|z| \leq r : r \in [0, 1)\}$. In other words, geometric series converges inside the disk of radius $R = 1$ and diverges outside this disk. It turns out that the situation is very similar for a general power series (14.1).

Proposition 14.3. *Define $R \in [0, \infty]$ by*

$$R = \sup\{r \geq 0 : (c_n r^n)_{n=0}^{\infty} \text{ is bounded}\}.$$

Then the power series (14.1) *converges absolutely and uniformly on any compact subset of the ball (disk) $B(z_0, R)$ and diverges at every point $|z - z_0| > R$.*

Remark 14.4. The constant R, quite naturally, is called the radius of convergence. The disk $B(z_0, R)$ is called the disk or region of convergence. Note that the proposition says nothing about the behavior of the power series on the boundary $\partial B(z_0, R)$. In short: anything can happen on ∂B.

Proof. The second part is immediate: if $|z - z_0| > R$ then by definition the terms of the series are not even bounded, therefore no convergence is possible. To prove the first part, take $\overline{B}(z_0, r)$, where $r < R$. Choose $r < \rho < R$, the definition of R implies that $(c_n \rho^n)$ is bounded: $|c_n|\rho^n < M$ for some constant M. Then for all $z \in \overline{B}(z_0, r)$ $|c_n(z - z_0)^n| \leq |c_n|\rho^n(r^n/\rho^n) \leq M(r/\rho)^n = M_n$. The series $\sum_{n \geq n} M_n$ converges as the geometric series with $0 < r/\rho < 1$, and therefore by the Weierstrass M-test the power series converges absolutely and uniformly on any $\overline{B}(z_0, r), r < R$ and hence for any compact subset of $B(z_0, R)$. ∎

Math 452: *Complex Analysis* by Artem Novozhilov©
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Remark 14.5. The given definition for the radius of convergence is difficult to use in practice. Its traditional formula is given by

$$R = \liminf_{n\to\infty} |c_n|^{-1/n},$$

if you know what lim inf is. In the textbook you can find a proof that if $(|c_n|^{-1/n})$ has a limit then R is this limit. Probably computationally the most convenient formula is

$$1/R = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|,$$

if this limit exists ($R = \infty$ if this limit is zero). This was discussed in Calc II as the ratio test.

For instance, recall that I defined

$$\exp z = e^z = 1 + z + \frac{z^2}{2!} + \ldots$$

Therefore I have

$$\lim_{n\to\infty} \frac{1}{(n+1)!}\frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

and hence $R = \infty$. The same infinite radius of convergence for the series of sine and cosine functions.

For the geometric series the ratio test will give, as expected, $R = 1$.

Remark 14.6. Slightly more can be proved. Specifically, the missing piece is that for $|z - z_0| < R$ the power series converges absolutely. See the textbook for the details.

Now I can apply the results about uniform convergence from the previous lecture to the power series. Specifically,

Proposition 14.7. *Let $B(z_0, R)$ be the disk of convergence of* (14.1) *with $R > 0$. Then*

1. *The power series converges to a continuous on $B(z_0, R)$ function.*

2. *For any $\gamma \subseteq B$*

$$\int_\gamma \sum_{n=0}^{\infty} c_n(z - z_0)^n \mathrm{d}z = \sum_{n=0}^{\infty} c_n \int_\gamma (z - z_0)^n \mathrm{d}z.$$

3. *If $\gamma \subseteq B$ is closed then*

$$\int_\gamma \sum_{n=0}^{\infty} c_n(z - z_0)^n \mathrm{d}z = 0.$$

4. *Series* (14.1) *is a holomorphic function inside $B(z_0, R)$.*

5.

6. *The power series can be differentiated term by term,*

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z - z_0)^{n-1},$$

for any $z \in B(z_0, R)$, and the radius of convergence of the last series if R again.

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof. The proof of the first two points follows from the fact the power series converges uniformly in each compact subset of $B(z_0, R)$ and any point z and any path γ can be considered inside some $\overline{B}(z_0, \rho)$, $\rho < R$. Point 3 is a direct calculation (recall the fundamental integral).

To prove point 4 I note that f is continuous as a limit of uniformly convergent series, and any integral of f along a closed γ is zero by point 3. By Morera's theorem f must be holomorphic.

To prove 5 I take $z \in B(z_0, R)$, $|z - z_0| < r < R$. Since f is holomorphic at z I can use Cauchy's integral formula to write ($\gamma = \partial B(z_0, r)$)

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^2} \mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{(w-z)^2} \sum_{n\geq 0} c_n (w-z_0)^n \mathrm{d}w = \sum_{n\geq 0} c_n \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{(w-z_0)^n}{(w-z)} \mathrm{d}w,$$

where I exchanged the order of summation and integration since power series converges uniformly.

Now note that if $f_n(w) = (w - z_0)^n$, then

$$f_n'(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{(w-z_0)^n}{(w-z)^2} \mathrm{d}w$$

on one hand, and $f_n'(z) = n(z - z_0)^{n-1}$ by direct calculations. Hence,

$$f'(z) = \sum_{n\geq 0} c_n n (z-z_0)^{n-1}$$

at least inside $B(z_0, R)$, where I performed all the manipulations. To see that the radius of convergence cannot exceed R note that the coefficients of $(z - z_0)f'(z)$ are nc_n, and clearly $|nc_n| > |c_n|$, hence the series for the derivative cannot converge at the points where the series for f diverges.

Finally, for 6, I note that $f(z_0) = c_0$. Invoking part 5, I get $f'(z_0) = c_1$. Continuing by induction I get the required formula. Note also that since we got formulas for the coefficients of a power series, they must be unique. Formally, if

$$f(z) = \sum c_k (z - z_0)^k$$

and

$$f(z) = \sum b_k (z - z_0)^k$$

then $c_k = b_k$. ■

14.2 Applications

The previous section was somewhat theoretical. Here I would like to consider a bunch of examples, mainly of the form: here is a given function. How to find its power series around given z_0. Ideally, I also would like to know the radius of convergence of this power series. Now we have enough tools and examples to make this procedure quite straightforward.

First I would like to note that the most direct way to find the power series, namely, to use the relation

$$c_n = \frac{f^{(n)}(z_0)}{n!},$$

is usually the most tedious one. So here is the rule: if it is possible to determine the power series without explicit use of the formulas above, go for it, save these formulas as the last resort.

Example 14.8. I know that

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}$$

if $|z| < 1$. Let me use this relation to find the power series of

$$f(z) = \frac{A}{z-a},$$

where A, a are some complex constants. First I note that if $z_0 = a$ then power series does not exist because the expression is not even determined at the point $z = a$. Next, let me write

$$\frac{A}{z-a} = \frac{A}{z - z_0 + z_0 - a} = -\frac{A}{(a-z_0)(1 - \frac{z-z_0}{a-z_0})}$$

Since the fraction can be represented, using the geometric series, as

$$\frac{1}{1 - \frac{z-z_0}{a-z_0}} = 1 + \frac{z-z_0}{a-z_0} + \left(\frac{z-z_0}{a-z_0}\right)^2 + \ldots,$$

I obtain the final result

$$\frac{A}{z-a} = -\frac{A}{a-z_0}\left(1 + \frac{z-z_0}{a-z_0} + \left(\frac{z-z_0}{a-z_0}\right)^2 + \ldots\right) = -\frac{A}{a-z_0} - \frac{A(z-z_0)}{(a-z_0)^2} - \ldots$$

This expression must be valid for all z

$$\left|\frac{z-z_0}{a-z_0}\right| < 1$$

or

$$|z - z_0| < |a - z_0|,$$

i.e., inside the disk with the center at z_0 and radius $|a - z_0|$.

Example 14.9. What is the power series of

$$\frac{1}{(1-z)^p}$$

around $z_0 = 0$? Here p is a natural number.

By differentiating the power series for the geometric series I find

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \ldots,$$
$$\frac{2}{(1-z)^3} = 2 + 3 \cdot 2z + 4 \cdot 3z^2 + \ldots,$$

and so on. Introducing

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

I end up with the general expression

$$\frac{1}{(1-z)^p} = 1 + \binom{p+1}{p} z + \binom{p+2}{p} z^2 + \binom{p+3}{p} z^3 + \ldots$$

The radius of convergence of this series is $|z| < 1$, because differentiating does not change it.

Example 14.10. Taking together two previous examples, I find

$$\frac{A}{(z-a)^p}=\frac{(-1)^pA}{(a-z_0)^p}+\binom{p+1}{p}\frac{(-1)^pA(z-z_0)}{(a-z_0)^{p+1}}+\binom{p+2}{p}\frac{(-1)^pA(z-z_0)^2}{(a-z_0)^{p+2}}+\dots$$

with the region of convergence $|z-z_0|<|a-z_0|$.

If the previous examples are clear it means that now the student should be able to represent any rational function as a power series around some z_0, the only condition is that z_0 is not a zero of denominator. Instead of general formulas, here is an example.

Example 14.11. Find the power series of

$$\frac{380+87z+3z^3}{(5-z)(2+z)(9+z^2)}$$

around $z=0$.

First, I note that this function can be written as (partial fraction decomposition)

$$\frac{380+87z+3z^3}{(5-z)(2+z)(9+z^2)}=\frac{5}{5-z}+\frac{2}{2+z}+\frac{20}{9+z^2}\,.$$

Now

$$\frac{5}{5-z}=\frac{1}{1-\frac{z}{5}}=\sum_{n\geq 0}\left(\frac{z}{5}\right)^n,$$

$$\frac{2}{2+z}=\frac{1}{1+\frac{z}{2}}=\sum_{n\geq 0}\left(\frac{z}{-2}\right)^n,$$

and

$$\frac{20}{9+z^2}=\frac{20}{9}\cdot\frac{1}{1+\frac{z^2}{9}}=\frac{20}{9}\sum_{n\geq 0}\left(\frac{z^2}{-9}\right)^n=\frac{20}{9}\sum_{n\geq 0}\left(\frac{z^2}{3\mathrm{i}}\right)^{2n}.$$

Therefore, I get the series with the coefficients

$$c_n=\begin{cases}\frac{1}{5^n}+(-1)^n\frac{1}{2^n}, & n=2m+1,\\ \frac{1}{5^n}+(-1)^n\frac{1}{2^n}+\frac{20}{9}(-1)^{n/2}\cdot\frac{1}{3^n}, & n=2m.\end{cases}$$

What is the radius of convergence? Note that the first "elementary" series converges for $|z|<5$, second for $|z|<2$, and third for $|z|<3$. Choosing the smallest our of these numbers I conclude that $R=2$ for my my series.

I chose here that $z_0=0$ but nothing (well, except that the calculations become a little more tedious) would change for a different z_0.

For instance, take $z_0=2$. Note that the radius of convergence is the smallest distance our of $|5-2|,|-2-2|$ and $|\pm 3\mathrm{i}-2|$, hence $R=3$. I will leave the rest of the details to the reader.

In the following examples I choose $z_0=0$ but note that no generality is lost here. If $z_0=a$ then let $w=z-a$ be a new variable, $f(z)=f(w+a)=\tilde{f}(w)$. Now I find the power series for $\tilde{f}$ around $w=0$. Returning to the original variable z I get the series around $z_0=a$.

Example 14.12.

$$f(z) = \frac{1}{1-z} + e^z.$$

using the known series

$$f(z) = (1 + z + z^2 + \dots) + (1 + z + \frac{z^2}{2!} + \dots) = \sum_{n\geq 0} \left(1 + \frac{1}{n!}\right) z^n.$$

The radius of convergence is 1, because the series for e^z converges everywhere in $\mathbf{C}$.

Example 14.13.

$$f(z) = (1 - z + z^2)e^z.$$

In a similar vein,

$$f(z) = (1 - z + z^2)(1 + z + z^2/2! + \dots) = \sum_{n\geq 0} \frac{z^n}{n!} - \sum_{n\geq 0} \frac{z^{n+1}}{n!} + \sum_{n\geq 0} \frac{z^{n+2}}{n!} = 1 + \sum_{n\geq 2} \left(1 - \frac{1}{n}\right) \frac{z^n}{(n-2)!}.$$

The radius of convergence is $R = \infty$.

Example 14.14.

$$f(z) = e^{-z^2}.$$

Using the series for the exponent, I get

$$f(z) = 1 - z^2 + \frac{z^4}{2!} - \frac{z^6}{3!} + \dots$$

Example 14.15.

$$f(z) = e^z \cos z.$$

All we need here is to remember the series for exp and cos:

$$f(z) = (1 + z + z^2/2! + \dots)(1 - z^2/2! + z^4/4! - \dots) = 1 + z - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \frac{1}{30}z^5 + 0 \cdot z^6 + \dots$$

Example 14.16.

$$f(z) = \arctan z.$$

Here I am taking the branch of the arctangent for which $\arctan 0 = 0$. If you recall that arctan can be represent through log, and

$$(\log z)' = \frac{1}{z},$$

for any branch, then one can show that

$$(\arctan z)' = \frac{1}{1+z^2}.$$

I know that

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots,$$

and the radius of convergence is $|z| < 1$. Hence,

$$\arctan z = \int_0^z \frac{\mathrm{d}w}{1+w^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots$$

The radius of convergence here is $|z| < 1$.

Example 14.17.

$$f(z) = \log(1+z)$$

Here I use the branch of log for which $\log 1 = 0$. I have

$$\log(1+z) = \int_1^{1+z} \frac{\mathrm{d}w}{w} = \int_0^z \frac{\mathrm{d}w}{1+w} = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$$

because

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$$

The radius of convergence is $|z| < 1$.

Example 14.18.

$$f(z) = \tan z.$$

Here is our first nontrivial example. So let me do the following trick: recall that $\tan z = \sin z / \cos z$, and the series for both cosine and sine we know. So, assuming that

$$\tan z = c_0 + c_1 z + c_2 z^2 + \ldots$$

I can write that

$$\sin z = (c_0 + c_1 z + c_2 z^2 + \ldots) \cos z$$

or

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = (c_0 + c_1 z + c_2 z^2 + \ldots)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right)$$

Comparing the coefficients at the same powers on the left and on the right, I find that

$$\begin{aligned} c_1 &= 1, \\ -\frac{1}{2}c_1 + c_3 &= -\frac{1}{6}, \\ \frac{1}{24}c_1 - \frac{1}{2}c_3 + c_5 &= \frac{1}{120}, \\ &\ldots \end{aligned}$$

which implies that $c_1 = 1, c_3 = 1/3, c_5 = 2/15, \ldots$

However, for this example we have no rigorous way to determine the radius of convergence (and our tests for it are of no help since we know no general formula for c_n). Yet. See the next lecture.